Accounting Principles from a Topological Point of View

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Some GAAP established more than six centuries ago still apply in accounting systems with high probability of success. However, in order to ensure these principles attain a maximum level of sufficiency, unbiasedness, efficiency, and consistency (SUEC), we need to improve their definitions giving them a robust universal conceptualization. This paper proposes a mathematical statement of seven fundamental accounting principles, related to a company's cash flow statements, describing p-entry accounting systems, for p=1,2,3,...,N. These systems are developed by means of equilibrium financial equations on topological spaces called simplicial complexes. Incidence algebraic structures as fundamental (co)homology groups are presented briefly to illustrate this new conceptualization.

INTRODUCTION

The human being has not yet completely unveiled the intangible laws governing the natural world. Undoubtedly, this is palpable in accounting and finance. However, while awaiting this unveiling, we feel distressed by the reality that science keeps developing at a frenetic pace although its development and application require an underlying set of concepts (assumptions and conditions) and conventions (usage and customs), definitions and principles drafted by a group of professionals. In the case of accounting and finance, these professionals are, in the US, the Financial Accounting Standards Board (FASB), which issues the Generally Accepted Accounting Principles (GAAP); or the International Financial Reporting Standards (IFRS) set up by the International Accounting Standards Board (IASB); or the Australian Government Company, known as the Australian Accounting Standards Board (AASB), which develops, issues and maintains accounting standards according to Australian law; or the Canadian IFRS Standards, developed and issued in the public interest by the International Accounting Standards Board (IAcSB) resulting from the 2015 Canadian GAAP (Canada-IFRS-Profile.pdf, 2016), just to name some.

On one hand, these principles have many positive functional properties such as improving the clarity of financial information communication, ensuring a minimum consistency level in a company's financial statements (CFS), introducing effectiveness to accounting operations, and facilitating the cross-comparison of financial information across different companies, among others.

Nevertheless, on the other hand, these principles are only a set of standards, norms and procedures that accountants must follow to ensure transparency in recording and reporting a company's accounting financial statements (Ross, 2000). In other words, accounting principles are rules based on assumptions, customs, usage and traditions for recording transactions (eBooks, Chap. 2). These principles do not provide much guarantee that CFS are free from errors or omissions intended to mislead investors. Furthermore, it is known that there are plenty of singularities within GAAP for unscrupulous independent bookkeepers and accountants, who may use these loopholes to falsify records in order to distort accounts (e.g., see CASA[1] (Haskin, 2016)). Moreover, most countries have their own accounting boards, where accounting standards are designed according to the specific requirements and laws of that country. Thus, GAAP differ among countries. For instance, there are differences[2] among the US's GAAP, Australia's AASB, and Canada's GAAP as they pertain to three different countries (see Endnotes).

How can we address these accounting weaknesses? We believe that accounting science must be aligned with related scientific disciplines and be based on axiomatic foundations of universal scope, as was focused in the earliest attempt to formulate accounting 'postulates' (Paton, 1922). Decades later, R. J. Chambers (1955) with his legacy "Blueprint for a theory of Accounting" and R.V. Mattessich (1957) with his work "Towards a General and Axiomatic Foundation of Accounting," and others, were concerned with the development of the axiomatic point of view, with their matrix-algebraic and set theoretical axiomatizations of accounting postulates. This time, we are not attempting to formulate accounting postulates, as they did. Instead, we have decided to improve and cultivate the development of more integrated modern accounting systems that provide SUEC-type information from their own rigorous analysis and synthesis in dual-aspect statics and dynamics.

This paper aims, firstly, to re-define the already well-known twelve or fifteen GAAP into fewer principles, here exactly seven, by regrouping them and providing them with mathematical support and emphasis to meet SUEC requirements. We emphasize the fact that there is a unique and specific oldest "backbone" accounting principle that has been applied to companies across the world for more than six centuries. It is the famous, 'great grandpa,' duality principle (DP). By means of this DP, general equilibrium financial equations of value are established from a stream of dated cash flows of financial transactions occurring at different times. Furthermore, we present a multidimensional *p*-entry accounting system for $p= 1, 2, 3, \ldots, N$ as a, probably pessimistic, conjecture. Secondly, our key objective, we provide a framework for GAAP, an underlying framework topological space, where these seven principles can act and their presence may be supported, and we explain the geometric implementation and algebraic group structures of these underlying spaces, called simplicial complexes. Incidence algebras obtained from a simplicial complex are used to analyze some applications of GAAP on these spaces. In particular, again, we emphasize the study of the duality principle.

In conclusion, we try neither to change already existing GAAP, nor to render their study more complicated. On the contrary, we want to give already existing studies a new representation focused on the mathematical point of view to provide GAAP with a more robust universal conceptualization. Such a conceptualization should be systematically included in a diachronic dimension throughout the evolution of accounting thought, in order to form a body of scientifically produced knowledge. With such a basis, the GAAP could be accepted by all the community, as well as by other sciences.

This paper is organized as follows: Section 2 provides a required literature review. Section 3 describes the proposed seven GAAP and the *p*-entry accounting system. Then, the supporting methodology is elaborated in Section 4. The study sample is provided in Section 5. Finally, we conclude with a summary of findings and recommendations in Section 6.

REQUIRED RELATED LITERATURE

Simplicial Complexes

In this section we present some standard instruments that will enable us to develop the main part of this paper. This set of tools consists primarily of the underlying space obtained by piecing together basic topological building blocks as collections of "triangles" called simplexes. Also, we present some geometrical properties and the essential algebraic invariant properties of the incidence algebra obtained on these topological spaces, such as the fundamental group, (co)homology groups, and others. More precisely, we use the topological invariants of the underlying Euclidean space $\mathbb{R}^N$ (it should be any other kind of space) to light up and describe the presence of the GAAP presented in the following section. Here, we consider the classical theory of simplicial complexes (Singer and Thorpe, 1967) in a concrete manner, pointing out only the elements required to achieve our goals.

Suppose $N \in \mathbf{Z}_+$ is sufficiently large. A set $\{v_0, v_1, \ldots, v_n\}$ of vectors in a vector space V is convex-independent if the set $\{v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0\}$ is linearly independent. Note that this definition does not depend on which vector is called v_0. If we look at the vectors $v_0, v_1, \ldots, v_n$ as $n+1$ points of $\mathbb{R}^N$ in general position and denote by $\Delta^n = (v_0, v_1, \ldots, v_n)$ the smallest convex set spanned by these points, we say that Δ^n is the $n-$simplex, that $v_0, v_1, \ldots, v_n$ are the vertices, and that n is the dimension of Δ^n. The j - simplex spanned by a subset $\{v_{i_1}, v_{i_2}, \ldots, v_{i_j}\}$ of $\{v_0, v_1, \ldots, v_n\}$ is called a face of Δ^n. In particular, Δ^n is its own face. We say that a collection of the faces $\partial\Delta^n$ of dimension less than n is the boundary of Δ^n.

Definition A. A simplicial complex $K = \{\Delta^n\}$ (Euclidean) is a finite family of simplexes of various dimensions in some $\mathbb{R}^N$ such that:

(1) If $\Delta^n \in K$, then every face of Δ^n is also in K.

(2) If $\Delta_1^m, \Delta_2^n \in K$, then $\Delta_1^m \cap \Delta_2^n$ is a face of each Δ_1^m and Δ_2^n.

The dimension of K is the maximum dimension of the simplexes of K. A simplicial complex is to be finite if $i \in I$ is finite, and is connected if for each couple of vertices (u, v) there are vertices $v_0, v_1, \ldots, v_n$ such that $v_0 = u$, $v_n = v$ and (v_{i-1}, v_i) is a simplex for each i in $\{1, 2, \ldots, n\}$. Note that since we assume that the simplicial complexes K are finite, their geometric realization $|K|$ exists and it is a subset of $\mathbb{R}^N$ and inherits the topology of $\mathbb{R}^N$.

Let K be a simplicial complex. An edge in K is an ordered pair $e = |v_1 v_2|$ of vertices of K, such that v_1 and v_2 lie in some simplex of K. v_1 is the origin of e, and v_2 is the end of e. If $e = |v_1 v_2|$, the edge $|v_2 v_1|$ is denoted by e^{-1}. An edge-path (EP) in K is a finite sequence $w = e_1 e_2 \ldots e_k$ of edges in K such that, for each $i \in \{1, 2, \ldots, k-1\}$, the end of e_i equals the origin of e_{i+1}. The origin of w is the origin of e_1, and the end of w is the end of e_k. Given two EPs $w = e_1 e_2 \ldots e_k$ and $\beta = e_1^{'} e_2^{'} \ldots e_m^{'}$ with the end of w equal to the origin of β, their product $w\beta$ is defined by $w\beta = e_1 e_2 \ldots e_k e_1^{'} e_2^{'} \ldots e_m^{'}$. The inverse of EP $w = e_1 e_2 \ldots e_k$ is EP $w^{-1} = e_k^{-1} e_{k-1}^{-1} \ldots e_1^{-1}$. An equivalence relation on the set of all EPs in K is defined as follows. If $e = |v_1 v_2|$ and $e' = |v_2 v_3|$ are such that v_1, v_2, v_3 are the vertices of a simplex, then the product ee' is edge-equivalent to the edge $|v_1 v_3|$. Two EPs w and β are edge-equivalent, denoted by $w \overset{E}{\sim} \beta$, if β can be obtained from w by a sequence of such elementary edge-equivalences. Edge-equivalence is an equivalence relation. Moreover, if w is an EP with origin v, then $ww^{-1} \overset{E}{\sim} |vv|$ Also, if $v_1, v_2, \ldots, v_k$ are vertices of a simplex, then $|v_1 v_2||v_2 v_3| \ldots |v_{k-1} v_k| \overset{E}{\sim} |v_1 v_k|$. If two equivalent EPs have the same extremities, the product just defined above, when it exists, is defined also on the equivalence classes, as well as on the set of edge-loops starting at a fixed point.

Theorem A (Singer and Thorpe, 1967, page 98). Let K be a simplicial complex, and let v_0 be a vertex of K. Let $E(K, v_0)$ be the set of edge-equivalence classes of EPs in K with origin v_0 and end v_0

(loops). Then $E(K, v_0)$ is a group, with identity $|v_0 v_0|$, under the operations of multiplication and inverse defined above for EPs. $E(K, v_0)$ is called the EP group of (K, v_0). The proof is a routine.

The EP group of a complex K is a purely "combinatorial" object; that is, it depends on only the vertices of K and those subsets which are vertices of a simplex. Its definition does not use the topological properties of the space $[K]$ (closed simplex). But, since we assume that all simplicial complexes are finite their geometric realization $|K|$ exists. Indeed, every simplicial complex determines an "abstract" simplicial complex, so, the EP group can then be defined for abstract complex. It is the same as an EP group of any realization of $|K|$. It is in this sense that we mean $E(K, v_0)$ is a purely combinatorial object. From this, we have the following theorem (Singer and Thorpe, 1967, page 98).

Theorem B. Let K be a simplicial complex, and let v_0 be a vertex of K. Then $E(K, v_0)$ is isomorphic with the fundamental group $\pi_1([K], v_0)$.

The proof is in Singer and Thorpe, page 98. Here, we only need its statement to continue our purpose. Notice that $\pi_1([K], v_0) = \{[\alpha] : \alpha$ is a closed EP, with v_0 as a based point$\}$, where $[\alpha]$ is an edge-equivalence class.

Incidence Algebras

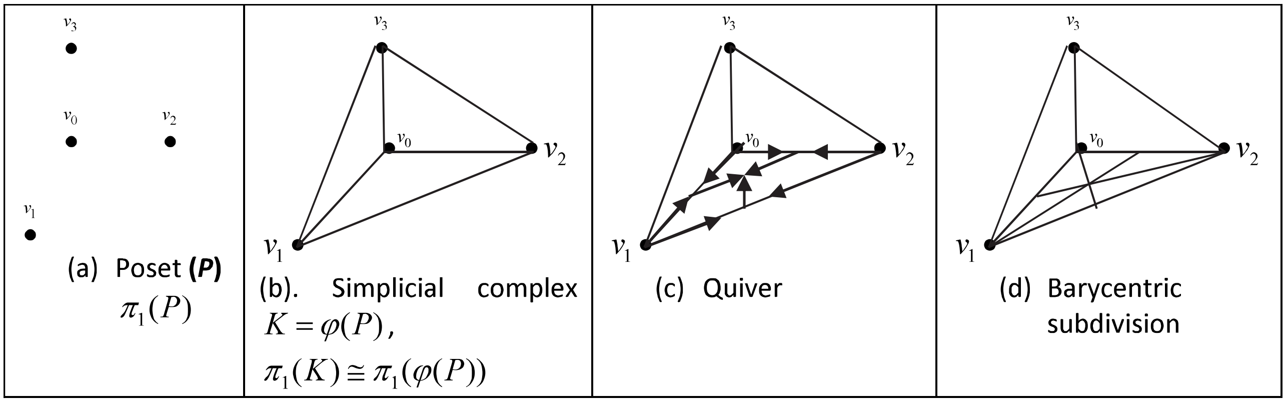
Incidence algebras, algebras obtained from a simplicial complex, were introduced by G.C. Rota (Rota, 1968). Here, the frame of our strategy begins with an underlying poset (P). Then, we focus on its associated simplicial complex (K), and define the incidence algebra $I_A(.)$ on it, to get algebraic structures. Namely, we want the algebraic fundamental group $\pi_1(K)$, which will be isomorphic to the topological fundamental group of the geometric realization of this simplicial complex, while the former is isomorphic to the algebraic fundamental group of its incidence algebra, $\pi_1(I_A(K))$. See Figure 1.

Let $(P, \leq)$ be a locally finite partially ordered set (poset) with a relation $\leq$, which will later be a set of business transactions. Let $Int(P)$ denote a set of nonempty intervals of P. That is, the sets $[x, y] := \{z \in P \mid x \leq z \leq y\}$ for all $x \leq y$. Let k be a field. With these ingredients, we provide a brief description of a simplicial complex, definition A above, from a poset point of view, as follows. To each poset P we associate a simplicial complex $\varphi(P) \approx K$, the set of non-empty simplexes Δ^n of K ordered by inclusion, where a n-simplex is a subset of P containing $n+1$ elements and totally ordered. The application φ is surjective but not injective. Let K be the above simplicial complex. Then, $|K(P)|$ is the geometric realization of the barycentric decomposition of K. For example, consider four points (very soon, four business transactions) v_0, v_1, v_2, v_3 that are not on the same plane, in $\square^3$. Let K be the set of non-empty parts of $\Delta^3 = (v_0, v_1, v_2, v_3) \subset \Delta^n$; its geometric realization is a tetrahedron, a 3-simplex or a four-entry accounting system in the next section. See Figure 3. Then, $K(P)$ contains all elements that are simplexes of K. Namely, one 3-simplex $\Delta^3 = (v_0, v_1, v_2, v_3)$, four 2-simplexes $\Delta_1^2 = (v_0, v_1, v_2)$, $\Delta_2^2 = (v_0, v_1, v_3)$, $\Delta_3^2 = (v_0, v_2, v_3)$, $\Delta_4^2 = (v_1, v_2, v_3)$, six 1-simplexes $\Delta_1^1 = (v_0, v_1)$, $\Delta_2^1 = (v_0, v_2)$, $\Delta_3^1 = (v_0, v_3)$, $\Delta_4^1 = (v_1, v_2)$, $\Delta_5^1 = (v_1, v_3)$, $\Delta_6^1 = (v_2, v_3)$, and four 0-simplexes $\Delta_i^0 = (v_{i-1})$, $i = 1, 2, 3, 4$. Their order is defined by $\Delta_i^0 \leq \Delta_j^1 \leq \Delta_k^2 \leq \Delta^3$ for all $i, j, k \in \{1, 2, 3, 4\}, i \neq j \neq k$ $i, j, k \in \{1, 2, 3, 4\}, i \neq j \neq k$ $i, j, k \in \{1, 2, 3, 4\}, i \neq j \neq k$. The associated quiver is drawn in the next figure (Figure 1). Then, $S(P(K))$ is the complex containing the total ordered subset of P. That is to say: for all $i, j, k \in \{1, 2, 3, 4\}$, the 0-simplexes are $\{\Delta_i^0\}, \{\Delta_j^1\}, \{\Delta_k^2\}, \{\Delta^3\}$, the 1-simplexes are

$\{\Delta_i^0,\Delta_j^1\},\{\Delta_i^0,\Delta_k^2\},\{\Delta_i^0,\Delta^3\},\{\Delta_j^1,\Delta_k^2\},\{\Delta_j^1,\Delta^3\},\{\Delta_k^2,\Delta^3\}, i\neq j\neq k$ the 2-simplexes are $\{\Delta_i^0,\Delta_j^1,\Delta_k^2\}$, $\{\Delta_i^0,\Delta_j^1,\Delta^3\},\{\Delta_i^0,\Delta_k^2,\Delta^3\}$, $\{\Delta_j^1,\Delta_k^2,\Delta^3\}$, $i\neq j\neq k$, and the 3-simplex is $\{\Delta_i^0,\Delta_j^1,\Delta_k^2,\Delta^3\}$.

The identification of a poset (a) and its associated ordered quiver (c), simplicial complexes (b) and their geometric realization (d) are visualized in the following figure (Figure 1).

FIGURE 1



A visualization of the geometric realization (incidence algebra) in a tetrahedron sitting on the first octant of $\mathbf{R}^3$ in which all three coordinates (X,Y,Z) are positive. (c) and (d) show one face only.

Theorem C. Let P be a poset, while the fundamental groups $\pi_1(P)$ and $\pi_1(K(P))$ are isomorphic.

A very good detailed proof that fundamental groups of finite and connected simplicial complexes are isomorphic was provided by Reynaud (Reynaud E., 2003).

We present now the definition of algebras obtained from the underlying spaces described above. These algebras are called incidence algebras of simplicial complexes.

Definition B. The **incidence algebra** $I_A(K(P))$ is the set of incidence functions $\varphi: Int(P)\rightarrow \mathrm{k}$, where, for our purpose, k will be a real □ -vector space with point-wise addition, subtraction and multiplication, and equipped with the convolution product $*$ as follows:

$$(\varphi*\psi)([x,y])=\begin{cases}\sum_{z\in[x,y]}\varphi([x,z])\psi([z,y]) & \text{for } x\leq y\\ 0 & \text{for } x>y\end{cases} \tag{1}$$

Note that the assumption of local finiteness is both necessary and sufficient for convolution to be well defined. This expression (3) in definition B satisfies the axioms of algebra (the multiplicative identity is the Kronecker delta function, there is a left/right/two sided convolution inverse $\varphi^{-1}([x,y])$, and $*$ is associative).

(Co)homology of Simplicial Complexes

Let K be a simplicial complex, and let G be the Abelian group of integers with addition operation. Let $C_l(K,G)$ denote the factor group of the free Abelian group generated by all oriented simplexes of K, modulo the subgroup generated by all oriented elements of the form $\langle v_0,v_1,v_2,\ldots,v_l\rangle+\langle v_1,v_0,v_2,\ldots,v_l\rangle$. Thus, $C_l(K,G)$, $l=0,1,2,\ldots$, is an Abelian group called the group of $l-$chains of K with integer coefficients.

Definition C. The boundary map $C_l(K,G) \xleftarrow{\partial} C_{l+1}(K,G)$ is the group homomorphism and the maps

$C_{l-1}(K,G) \xleftarrow{\partial} C_l(K,G) \xleftarrow{\partial} C_{l+1}(K,G)$ satisfy $\partial^2 = \partial \circ \partial = 0$. See the proof in Singer-Thorpe (Singer-Thorpe, pp. 155-156).

Definition D. Given K and G as above, the group $H_l(K,G) = \dfrac{Z_l(K,G)}{B_l(K,G)}$ is called the *l*th homology group of K with coefficients in G, where the elements of $Z_l(K,G) = [c \in C_l(K,G); \partial c = 0]$ are called cycles and will geometrically be a "chain'' of *l*-simplexes without boundary and the elements of $B_l(K,G) = [\partial c; c \in C_{l+1}(K,G)]$ are called boundaries of (*l*+1)-simplexes.

It turns out that the groups $H_l(K,G)$ depend only on the topology of $[K]$. See examples in Singer-Thorpe, Chapter 6. Similarly, one defines cohomology.

PROPOSED SEVEN GAAP AND THE *p*-ENTRY ACCOUNTING SYSTEM

Now, we consider the current US GAAP, without loss of generality, and proceed to regroup them into just seven accounting principles, "The Seven Commandments C_c of Accounting¨ where $c = 1, 2, ..., 7$, which will be very beneficial for student learning, as we show later. Specifically, these proposed seven principles are the following:

- **C_1. Entity-Going Concern (EGC).** Means there is a business entity1, that is separate from its owner(s), and the life of the business entity would continue infinitely long and will never dissipate.
- **C_2. Duality Principle (DP).** It is the particular case of the *p*-entry accounting system, when *p*=2, which is an equation established with the participation of debtor and creditor in the financial transactions carried out by a business entity.
- **C_3. Historical Cost - Measurement (HCM).** Transactions of business are recorded at their original cost and date in standard units of measure of items that can be measured and quantifiable in terms of money.
- **C_4. Materiality-Estimate-Conservatism (MEC)**. The accountant is allowed to round off tiny values, estimate and minimize the expected error and agree more with an understatement than an over-evaluation.
- **C_5. Consistency and Periodicity (CP).** Each individual enterprise must choose a single method of accounting and reporting consistently over an accounting period of business, normally one year.
- **C_6. Substance over Form (SOF).** The entity accounts and presents in the financial statements for items according to their substance and economic reality and not just its form.
- **C_7. Completeness (C).** The company's data disclosure in a particular period, in the company financial statements, must be sufficient, unbiased, efficient, and consistent (SUEC) for decision making.

The principle of completeness deserves a brief explanation. The role of accounting and statistics are very similar in practice. Both begin with collecting and recording data, then organizing, processing and presenting information for decision making. Then, reporting accounting financial statements (RAFS) should satisfy various statistical properties of estimators of recorded data to decide which estimator is most appropriate in a given situation. That is, which RAFS will expose us to the smallest risk, which will give us the most real information at the lowest cost and free of errors. We believe the RAFS that satisfies the seven commandments. Hence, accounting statements that follow the GAAP must be SUEC.

Proposition A: RAFS are SUEC if and only if they satisfy conditions (i) to (iv) below:

(i)- Sufficiency: RAFS utilize all the information from each individual sample period relevant to the estimation of the RAFS accounts of remaining business periods in the entity.
(ii)- Unbiasedness: The value expected for each period sample for any variable of the RAFS should be at least or equal to its total real value (parameter).
(iii)- Efficiency: The information contained in the RAFS should have minimum variability. The smaller the variance, the greater the information (Miller & Miller, 1999, page 327).
(iv)- Consistency: For a large number of business periods, *n*, during the entity existence, the estimators will take on values very close to their respective real values of the RAFS.

The *p*-entry Accounting System for $p = 1, 2, 3, \ldots, N$

In this section, we explore and propose a construction of a general theory of accounting systems. That is, taking the underlying complexes described above, as a landscape, we spread on it the seven regrouped accounting principles, as though sowing seed on farmland, and analyze their behavior on these spaces. Also, we recall a little accounting history, to establish a better scheme and systemic description of scientifically accurate accounting systems.

We assume that our simplicial complex K is nonempty. If $K = \phi$, the dimension of K is -1. This means that accounting principle C_1 fails. That is, there is no entity business. Consequently, there are no business transactions, even though the empty ϕ is a face of every simplex and thus belongs to K, by condition (1) of definition A. For this reason, we begin with the 0-dimensional simplexes, the set of vertices of a simplicial complex, which represent the 0-dimensional transaction (points are transactions with no flows). It is well known that for *p*=1, there is a *1*-entry accounting system, called a single entry system, in which only one aspect of a transaction is recorded. For instance, if a sale is made to a customer, only sales revenue will be recorded. However, the other extreme of the transaction relating to the receipt of cash or the granting of credit to the customer is not recognized. We believe that this is one of the reasons why this system has been superseded by 1-dimensional transactions or the well-known $(p = 2)-$ entry accounting system, explained next.

The 2-entry accounting system or $\mathbf{C_2}$ principle above is the ancestor **Duality Principle (DP)** and was devised to account for more aspects of a transaction (L. Pacioli, 1494). This principle is established in several ways, but all with the same purpose and meaning. Here, we adopt a mathematical point of view following R. Mattessichs' (1957) ideas that an account is an ordered pair (X, Y) of nonnegative elements where the first variable is debits and the second is credits. So, for any values of an account (X, Y) we have equivalent relations:

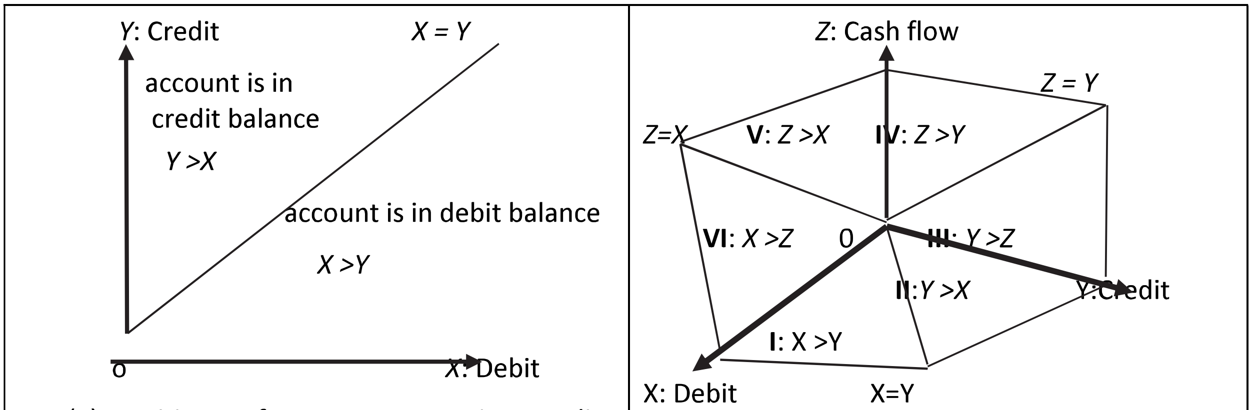
$$(X,Y) \equiv (X - Y, 0) \equiv (X - Y > 0, 0), \text{ if } X > Y \text{ the account is in the debit balance category} \quad (2)$$

$$(X,Y) \equiv (0, Y - X) \equiv (0, Y - X > 0), \text{ if } Y > X \text{ the account is in the credit balance category} \quad (3)$$

$$(X,Y) \equiv (X - Y = 0, Y - X = 0) \equiv (X = Y, Y = X), \text{ if } X = Y \text{ the account is balanced} \quad (4)$$

Note that the latter case is the C_2 principle, the Duality Principle (debits = credits). On this scenario, we can represent the set of all states of business transactions. Specifically, on the simplicial space located at the first quadrant of the Euclidean plane $\mathbf{R}^2$ where $N = 2$ and the axes are (X, Y). See Figure 2(a), which also shows the set of all possible transactions that can be represented as a transition matrix whose secondary diagonal contains the set of all values of the balanced accounts (4), in a horizon of business, where the debits (2) and credits (3) values are below or above the diagonal, respectively.

FIGURE 2



(a) Position of an account in Duality Principle, which could also be two accounts.

(b) Positions of an account, which could also be three accounts.

(2a) Account with two extremes, ordered pair (X>0, Y>0), with an infinite number of transition states of an account on 1-simplex. (2b) Account with three extremes, a triad (X>0, Y>0, Z>0), with three 2-simplexes, namely, $X\hat{0}Y, Y\hat{0}Z, Z\hat{0}X$ with the regions (I,II), (III,IV), (V,VI), respectively.

Similarly, since accounting is a set of accounts, we can extend and apply this procedure to three accounts, $(p=3)-$ entry accounting system. Let us say that we have a triple (X,Y,Z) in $\mathbf{R}^3$, which are accounts with three variables or could be a set of three accounts, each with nonnegative values, with X and Y as above and Z ,WLOG, is the amount of cash flow (CF) in the company in the same period of business date of X and Y. In this moment, we are in the 2-dimensional simplex (subset of $\mathbf{R}^3$) and 2-dimensional transactions and 3-entry accounting system (X,Y,Z) on some business entity with the same periods. Notice that we added one more account to the duality principle (see Figure 2(b)). So, from this underlying space we produce the following combinatorial[2] relations between these accounts:

$$(X,Y,Z)\equiv(0,Y-X,Z-X)\equiv(0,Y-X>0,Z-X>0)\equiv(0,Y>X,Z>X) \text{ and } \frac{Y+Z}{2}>X \quad (5)$$

These equivalences show that company accounts have no debit and the sum of credit and CF is greater than debit. Thus, the average of credit and CF is greater than debit, which means that the account of the entity is in the credit balance.

$$(X,Y,Z)\equiv(X-Y,0,Z-Y)\equiv(X-Y>0,0,Z-Y>0)\equiv(X>Y,0,Z>Y) \text{ and } \frac{X+Z}{2}>Y \quad (6)$$

These equivalences show that the company accounts do not have enough credit and the sum of debit and CF is greater than credit. Thus, the average of debit and CF is greater than credit, which means that the account of the entity is in the debit balance.

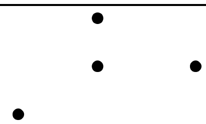
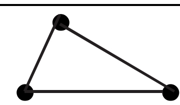
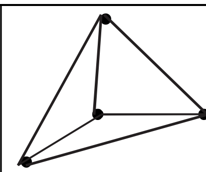
$$(X,Y,Z)\equiv(X-Z,Y-Z,0)\equiv(X-Z>0,Y-Z>0,0)\equiv(X>Z,Y>Z,0) \text{ and } \frac{X+Y}{2}>Z \quad (7)$$

Similarly, these equivalences show that the company accounts do not have sufficient CF, but the sum of debit and credit is greater than CF. Thus, the average of debit and credit is greater than CF. From these, we can deduce that the account of the entity is in the credit balance zone, if credit is greater than debit (5). Likewise, the account of an entity is in the debit balance zone, if debit is greater than credit (6).

Now, if we recall the C_2 principle, $X = Y$, and replacing it on the right side of relation (7) we get that either $X > Z$ or $Y > Z$. This means, again, that the account is either in the debit or credit zone, respectively. That is, a 2-entry accounting system is a subaccount of a 3-entry accounting system. Therefore, we believe that accounting systems may be generalized to p-entry accounting systems. So, one may take the collection of all sets of accounts and set up an equivalence relation by means of inclusion of accounts, like a truckload full of gravel. At this point we preach, probably pessimistically or perhaps optimistically, the existence of a multidimensional *p*-entry accounting system for *p= 1, 2, 3,…, N*, as a multidimensional simplicial complex already exists. This is a conjecture requiring further research.

We can summarize this explorative initial development by geometrically illustrating the generalization of the accounting systems in the following figure (Figure 3).

FIGURE 3

 0-dimensional simplex	Left debit = right credit 1-dimensional simplex	 2-dimensional simplex	 3-dimensional simplex	…..
(p=1)-entry accounting system	***(p***=2)-entry accounting system	***(p*=3)**-entry accounting system	***(p***=4)-entry accounting system	…..
0-dimensional transaction	1-dimensional transaction	2-dimensional transaction	3-dimensional transaction	…..

Relationship between simplicial complexes, *p*-entry accounting system and business transaction.

This way, these accounting principles are fulfilled by the rules of logic and by the rules of linguistic symptomatology in its three aspects, symbols, semantics and syntax (construction of information), and the fundamental Duality Principle C_2. When computing this, it provides us with equivalence relations on the value movement of the assets and a backbone of an inductive generalization of the rest of p-entry accounting systems, for $p = 1, 2, 3, ..., N$, acting on the underlying simplicial complex space.

We conclude this section stating that accounting systems are a set of accounts lodged by inclusion, as a simplicial complex is lodged by its dimension.

SUPPORTING METHODOLOGY AND DEVELOPMENT

Let us recall the first **C_1** commandment (EGC) from a mathematical point of view. It tells us that the simplicial complex $K \neq \phi$. This means, for large enough N, there is at least one business entity existing in the space $\mathbb{R}^N$, which never vanishes (as long as the owner wishes). In particular, let $E_1, E_2, ..., E_n$ in $\mathbb{R}^N$ be a collection of maximum expenses that this entity can spend on each $ith-$period of a sequence of business transactions (years, months, days, etc.). For our purpose, these expenses can be written as vectors that are $v_1 = (E_1, 0, ..., 0), v_2 = (0, E_2, ..., 0), ..., v_n = (0, 0, ..., E_n)$, which very soon will be the vertices (0-dimensional simplexes of expenses) of the N - dimensional simplicial complex. To be more precise, we focus on the case $N \leq n$, where the number of business transactions is greater or equal than the dimension of the simplicial complex. Similarly, for each expense $E_i, i = 1,2,3,....,n$ on the $ith-$ period, let $I_1, I_2, ..., I_i, ..., I_n$ be a collection of income of the same entity corresponding to the same business period. That is, there are ordered pairs (E_i, I_i) sitting on their corresponding vectors v_i and $v_i^{'}$, respectively. Here too, vectors $v_1^{'} = (I_1, 0, ..., 0), v_2^{'} = (0, I_2, ..., 0), ..., v_n^{'} = (0, 0, ..., I_n)$ are the 0-dimensional simplexes denoting an entity's (should be n business entities) income in n business periods.

An unbroken landscape of all of this vector space action on the same simplicial complex can be seen, for $N = 3$, in Figure 4.

Now, from Figure 4, it is easy to see the geometry of these two sets, $\{v_i\}_{i=1}^{n}$ and $\{v_i^{'}\}_{i=1}^{n}$, of vectors, each set in a general position that will very soon give us two hyper-planes of income and expenses, respectively. To formalize this methodology, we establish the following proposition.

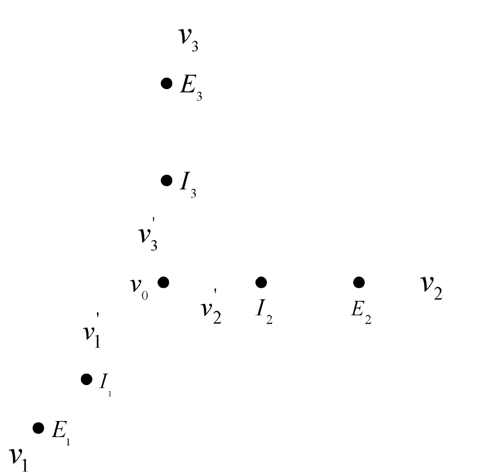
Proposition B: (i) If $v_i^{'} \subseteq v_i$, then $I_i \leq E_i$ for all $i = 1,2,...,n$.

(ii) If $v_i \subseteq v_i^{'}$, then $E_i \leq I_i$ for all $i = 1,2,...,n$.

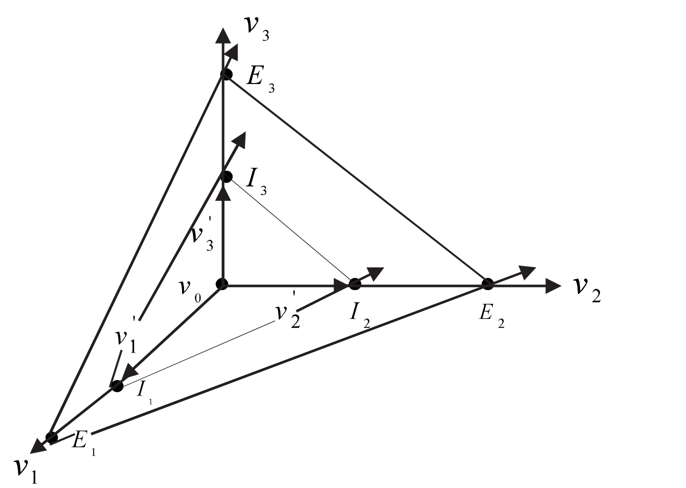
Part (i) of this proposition means the business entity transactions are in the debit zone (borrowers with income lower than expenses). That is, the company's expenditures exceeded its income. Part (ii) means the business transactions are in the credit zone (lenders with income higher than expenses). If, moreover, from (i) and (ii), $|v_i| = |v_i^{'}|$, then $E_i = I_i$ for all $i = 1,2,3,...,n$. This is the C_2 accounting principle. It means that the expenses hyperplane coincides with the income hyperplane.

Proof of (i). At this stage, we want to construct two hyperplanes lodged in the geometrical realization of $|K|$. The first hyperplane is called expenses plane (P_E) and the second plane is called income plane (P_I) or it could also be the plane of revenue, investments, loans, credits, etc. For the first hyperplane, we start with vectors $v_0, v_1, ..., v_n$, as represented above, where $v_0 = (0,0,...,0)$. Intuitively, we are already acting on the simplicial complex K.

FIGURE 4



A discrete 3-simplicial complex. $E_i, I_i, v_i, v_i^{'}$ are as described above.



Two hyperplanes lodged in the simplicial complex $\mathbf{R}^3 = K$. The expenses plane $P_E := E_1E_2E_3$ and the income plane $P_I := I_1I_2I_3 \to P_I \subseteq P_E$.

View of expenses, income, and vector action on the simplicial complex in $\mathbf{R}^3$.

Let S= $\{v_2 - v_1, v_3 - v_1, ..., v_n - v_1\}$ be vectors in $\mathbb{R}^N$ in general position, such that they can be written as entries in an array of order $n \times n$. An interesting special matrix results from this development:

$$M_n = \begin{pmatrix} k_1 & k_2 & & \dots & k_{n-1} & k_n \\ -E_1 & E_2 & & \cdots & 0 & 0 \\ -E_1 & 0 & E_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -E_1 & 0 & 0 & \cdots & E_{n-1} & 0 \\ -E_1 & 0 & 0 & \cdots & 0 & E_n \end{pmatrix} \tag{8}$$

where E_i is the ith period-expenses for $i = 1, 2, \cdots n$. On the first row, $k_1, k_2, ..., k_{n-1}, k_n$ are the unit standard vectors. Each matrix row is obtained from set S. For example, for the second row $v_2 - v_1 = (0, E_2, 0, 0, ..., 0) - (E_1, 0, 0, ..., 0) = (-E_1, E_2, 0, 0, ..., 0)$.

In this moment, we are at principle C_5 . The Euclidean simplicial complex of dimension N is the entity with n periods of business transactions, where we can consistently repeat this treatment for $N = 1, 2, 3...$ inductively over a business entity's accounting period.

We now compute the value of E_i by either adding or subtracting a factor, depending on the location of E_i on the dated cash flow and the time value of money at a rate r . For instance, the value equation is:

$$E_i = I_1(1+r)^{i-1} + I_2(1+r)^{i-2} + \cdots + \frac{I_{n-1}}{(1+r)^{n-i-1}} + \frac{I_n}{(1+r)^{n-i}} \quad \text{for } i = 1, 2, 3, ..., n \tag{9}$$

Thus, we compute the determinant $|M_n|$, which is the normal vector to the plane that we expect. With the help of MAPLE we can easily find the direction numbers of this normal:

$$|M_n| = \left(\prod_{i=2}^{n} E_i\right)k_1 + \left(\prod_{\substack{i=1 \\ i\neq 2}}^{n} E_i\right)k_2 + \cdots + \left(\prod_{\substack{i=1 \\ i\neq n-1}}^{n} E_i\right)k_{n-1} + \left(\prod_{\substack{i=1 \\ i\neq n}}^{n} E_i\right)k_n \tag{10}$$

Hence, using the results (9) and (10) the dynamic hyperplane of maximum expenses (P_E) containing the point $(E_1, 0, 0, \cdots, 0)$ and featuring the normal vector $\left(\prod_{i=2}^{n} E_i, \prod_{\substack{i=1 \\ i\neq 2}}^{n} E_i, \cdots, \prod_{\substack{i=1 \\ i\neq n}}^{n} E_i\right)$ can be expressed by the following equation in a standard form:

$$P_E := \sum_{j=1}^{n} \prod_{\substack{i=1 \\ i\neq j}}^{n} E_i (X_j - x_j) = 0 \tag{11}$$

where X_j plays the same role as the vectors v_j for $j \in \{1,2,...,n\}$ and $0 \le x_j \le E_j$. Clearly, we conclude that $P_I \subseteq P_E$, since for each $i = 1,2,3,...,n$, $I_i \le E_i$. See Figure 4.

Proof of (ii). Let $v_0^{'}, v_1^{'}, ..., v_n^{'}$ be vectors in general position, described above, where $v_0^{'} = (0,0,...,0)$ Then, since each $v_i \subseteq v_i^{'}$ and $E_i \le I_i$, the income Cartesian hyperplane P_I containing P_E and the point $(I_1, 0, 0, \cdots, 0)$ and featuring the normal vectors $\left(\prod_{i=2}^{n} I_i, \prod_{\substack{i=1 \\ i \ne 2}}^{n} I_i, \cdots, \prod_{\substack{i=1 \\ i \ne n}}^{n} I_i \right)$ can be expressed by the following equation in standard form:

$$P_I := \sum_{j=1}^{n} \prod_{\substack{i=1 \\ i \ne j}}^{n} I_i (X_j^{'} - x_j) = 0 \tag{12}$$

where $X_j^{'}$ plays the same role as the vectors $v_j^{'}$ for $j \in \{1,2,...,n\}$ and $0 \le x_j \le I_j$. The procedure to complete the proof is the same as the one for equation (11). *Hint*: the reader should draw a figure equal to Figure 4, then interchange v_i with $v_i^{'}$ and E_i with I_i for all i and conclude that $P_E \subseteq P_I$.

One notices that the animated two hyperplanes[3] move randomly up and down, while eventually the financial statement of an entity is in the debit or credit zone, and both hyperplanes meet instantaneously the moment the company expenses equal its income. Hence, the Duality Principle C_2 appears again. That is,

$$\text{If } P_I \subseteq P_E \text{ and } P_E \subseteq P_I \Rightarrow P_I = P_E \tag{13}$$

It means that the business entity exists and its expenses and income are equally likely, in all business periods (principles $\mathbf{C_1}$ and $\mathbf{C_5}$).

THE STUDY SAMPLE

Here we continue using the already established notations and all instruments presented above as fundamental frame. Let C be a representative company in a perfect financial market whose annual gross income (AGI) (in millions of US$) occurs according to Rule $I_i = 2 + 1.5^i$, where $i = 1,2,3,...,n = N$ is the dimension of the Euclidean simplicial complex (ESC) (or N-dimensional transaction). Let us consider an arbitrary $p-$entry accounting system. For example, it could be the 3-entry accounting system (X,Y,Z), the stronger cousin of the dual accounting principle, which involves three elements (e.g., deposit, withdrawal and cash flow) or any other triple, described above. Here, the ESC is a closed orthant immersed in $\mathbf{R}^N$ that constrains all coordinates to be positive. This orthant looks like a half-open book where each page is a 2-simplex.

Now, varying the values of i in rule $I_i = 2 + 1.5^i$, we get C's AGI for each business year period and, with these results, using formula (9), we compute the maximum expenses E_i that the company C can afford in each business period. That is, we show some computations considering a bank lending average rate of $6.65\% \approx 7\%$, without loss of generality, in the United States.

$I_1 = 2 + 1.5^1 = 3.5$

$$E_1 = 3.5 \times 1.07^0 + \frac{4.25}{1.07^1} + \frac{5.375}{1.07^2} + \frac{7.0625}{1.07^3} + \frac{9.59375}{1.07^4} + \frac{13.3906}{1.07^5} = 34.7981, \text{ then}$$

$v_1 = (34.7981, 0, 0, 0, 0, 0)$, and

$$\prod_{i=2}^{6} E_i = E_2 E_3 E_4 E_5 E_6 = 37.2340 \times 39.8404 \times 42.6292 \times 45.6133 \times 48.8062 = 140778734.1311$$

We complete the computations for $i = 1, 2, 3, 4, 5, 6$ and place the results in the interesting matrix constructed in (8) and in Table 1.

$$M_6 = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ -34.7981 & 37.2340 & 0 & 0 & 0 & 0 \\ -34.7981 & 0 & 39.8404 & 0 & 0 & 0 \\ -34.7981 & 0 & 0 & 42.6292 & 0 & 0 \\ -34.7981 & 0 & 0 & 0 & 45.6133 & 0 \\ -34.7981 & 0 & 0 & 0 & 0 & 48.8062 \end{pmatrix}$$

TABLE 1
SET OF INCOME AND EXPENSES

Year i	I_i	E_i	v_i	$\prod_{\substack{i=1 \\ i \neq j}}^{6} E_i$, $j \neq row$	v_i'
1	3.5	34.7981	(34.7981,0,0,0,0,0)	140778734.1311	(3.5,0,0,0,0,0)
2	4.25	37.2340	(0,37.2340,0,0,0,0)	131568793.7951	(0,4.25,0,0,0,0)
3	5.375	39.8404	(0,0,39.8404,0,0,0)	122961427.8011	(0,0,5.375,0,0,0)
4	7.0625	42.6292	(0,0,0,42.6292,0,0)	114917297.7247	(0,0,0,7.0625,0,0)
5	9.59375	45.6133	(0,0,0,0,45.6133,0)	107399211.8125	(0,0,0,0,9.59375,0)
6	13.3906	48.8062	(0,0,0,0,0,48.8062)	100373158.9054	(0,0,0,0,0,13.3906)
⋮	⋮	⋮	⋮	⋮	⋮

Set of income and expenses (I_i, E_i), $i = 1, 2, ..., 6$, and normal vectors of 5-dimensional plane in $\mathbf{R}^6$, which contains an immersed closed orthant, the business transaction environment.

Hence, from (11), we have the equation of entity C's expenses hyperplane, as follows,

$$P_E := \sum_{j=1}^{6} (v_j - x_j) \prod_{\substack{i=1 \\ i \neq j}}^{6} E_i = 0, \quad 0 \leq x_1, x_2, x_3, x_4, x_5, x_6 \leq E_i, \text{ for each } i \text{ and } n = N = 6 \qquad (14)$$

On this plane, we can interpret the occurrence of infinitely many business transactions depending on the zone they are located on. That is, they occur randomly on any zone of the business transition state hyperplane.

Let us be more concrete presenting a simpler case for $n = N =$ 3-dimensional transaction (3-dimensional simplex or 4-entry accounting system). See figures 3 and 4.

Again, using equation (9), we place the results of computations for $i = 1, 2, 3$ in Table 2.

TABLE 2
SET OF INCOME AND EXPENSES

Year i	I_i	E_i	v_i	$\prod_{\substack{i=1 \\ i \neq j}}^{3} E_i$, $j \neq row$	$v_i^{'}$	$\prod_{\substack{i=1 \\ i \neq j}}^{3} I_i$, $j \neq row$
1	3.5	12.1667	(12.1667, 0, 0)	181.3424	(3.5, 0, 0)	22.8438
2	4.25	13.0184	(0, 13.0184, 0)	169.4785	(0, 4.25, 0)	18.8125
3	5.375	13.9297	(0, 0, 13.9297)	158.3910	(0, 0, 5.375)	14.8750

Set of income and expenses $(I_{i,} E_i)$, $i = 1, 2, 3$ and normal vectors of 2-dimensional plane in $\mathbf{R}^3$, which contains an immersed closed octant, the environment of all business transactions.

We replace the data of Table 2 in (11) and, simplifying, we obtain:

$$\sum_{j=1}^{3}(v_j - x_j)\prod_{\substack{i=1 \\ i \neq j}}^{3} E_i = (v_1 - x_1)E_2E_3 + (v_2 - x_2)E_1E_3 + (v_3 - x_3)E_1E_2 = 0$$

$$= 6619.0166 - 181.4324x_1 - 169.4785x_2 - 158.391x_3 \text{ for } 0 \leq x_j \leq E_j \text{ and } j = 1, 2, 3. \quad (15)$$

This comes from:

```
> simplify((<12.1667,0,0>-<x[1],0,0>)*181.3424+(<0,13.0184,0>-
<0,x[2],0>)*169.4785+(<0,0,13.9297>-<0,0,x[3]>)*158.3910)=0;
```

$$\begin{bmatrix} 2206.338578 - 181.3424000\, x_1 \\ 2206.338904 - 169.4785000\, x_2 \\ 2206.339113 - 158.3910000\, x_3 \end{bmatrix} = 0$$

Next, we relabel the expenses vectors v_j with the real numbers E_j and carry out the operations, obtaining the result shown in (15).

Similarly, from the data in Table 2, we have a 2-plane of income:

$$P_I := \sum_{j=1}^{3}(v_j^{'} - x_j)\prod_{\substack{i=1 \\ i \neq j}}^{3} I_i = (v_1^{'} - x_1)I_2I_3 + (v_2^{'} - x_2)I_1I_3 + (v_3^{'} - x_3)I_1I_2 = 0 \quad (16)$$

```
>simplify((<3.5,0,0>-<x[1],0,0>)*22.8438+(<0,4.25,0>-
<0,x[2],0>)*18.8125+(<0,0,5.375>-<0,0,x[3]>)*14.8750)=0;
```

$$\begin{bmatrix} 79.95330000 - 22.84380000\, x_1 \\ 79.95312500 - 18.81250000\, x_2 \\ 79.95312500 - 14.87500000\, x_3 \end{bmatrix} = 0$$

$$P_I := \sum_{j=1}^{3}(v_j^{'} - x_j)\prod_{\substack{i=1\\ i\neq j}}^{3} I_i = 239.85955 - 22.8438\, x_1 - 18.8125\, x_2 - 14.875\, x_3 = 0 \quad (17)$$

Of course, in these two scenarios the company owner can at any time request loans and grant loans at the cost of the valuation of the equilibrium interest rate in the market. Also, when $P_E = P_I$, the C_2 accounting principle instantaneously shows up again. For the simplest case of this sample study, see Ross (2000), Chapter 3.

CONCLUSIONS AND RECOMMENDATIONS

This research study concludes that it is an emergency in the accounting science to ensure a maximum level of sufficiency, unbiasedness, efficiency, and consistency (SUEC) for GAAP, giving them a robust conceptualization from a mathematical point of view based on axiomatic foundations of universal scope. Hence, we propose and document the current US GAAP, without loss of general nature, and proceed with regrouping them into just seven accounting principles, "The Seven Commandments C_c of Accounting," where $c = 1, 2, ..., 7$, which will be very beneficial to student learning. This regrouping was done by similarity, after we explored the principles' behavior and roles in topological spaces called simplicial complexes. Note that we do not try to change already existing GAAP. Instead, we want to offer a more general approach to existing GAAP, using a mathematical approach (e.g., Paton, 1922; Chambers, 1955; Mattessich, 1957). We need GAAP without the name of the country.

Also, in this study, we take advantage of the landscape-like nature of a simplicial complex and develop, explore and propose the construction of a general *p*-entry accounting system theory. That is, taking the abovementioned underlying complexes as framework, we spread on it the seven regrouped accounting principles, as though sowing seed on farmland, and analyze their growth behavior on these spaces with a development and methodology supporting the construction of dynamic hyperplanes, e.g., income and expenses hyperplanes, supported by Proposal B and its study sample; concluding with a probably pessimistic or perhaps optimistic conjecture. We establish this conjecture as follows: since accounting is a set of accounts and the simplest set is a 1-entry accounting system (1EAS) superseded by a 2-entry accounting system (2EAS), and so on, one can establish an equivalence relation by including accounts, like a truckload full of gravel, i.e., $1EAS \subseteq 2EAS \subseteq .... \subseteq pEAS$, as a simplicial complex is usually constructed as shown in Figure 3. Hence, we conclude that the accounting multiple system exists in the real world, although it may not exist in the mind of accountants.

Additionally, this study opens lines of debate and research regarding the importance of a generalization of accounting systems theory. It also suggests that it would be a mistake to stop here. Instead, it urges to conduct an exploratory study of algebraic groups of incidence algebras and simplicial complexes. However, this work is in progress.

Finally, we believe that we are exploring a mysterious area of accounting science where one can find fruitful responses that will help analyzing the behavior of GAAP from this new perspective, on topological spaces. Furthermore, we notice that all business transactions should occur continuously and randomly represented on each face of a simplicial complex, viewed like the pages of a half-open book (see Figure 4). This means the space underlying the set of all transactions is a dynamic, random and complete financial market. So, further studies should also focus on quantized financial market models that have spaces underlying simplicial complexes (Huarca, 2017).

ENDNOTES

1. An unfortunate situation such as the CASA case could be helpful as an important example for accounting professionals and accounting students.
2. Difference between US GAAP and Canadian GAAP, January 13, February 23 2016 In "Business".
3. Also known as ¨Double Entry Principle¨, but here we call 2-entry accounting system.
4. In this paper, a business entity means a company, enterprise, university, an investor, any financial institution, a person, etc.
5. We leave intentionally the analysis of other combinatorial possibilities of these three accounts (X, Y, Z)
6. Actually, there are infinitely many hyperplanes, one for each instantaneous state of the business in the entity, moving according to the song: "Income is greater than expenses and/or are less than expenses, but they converge instantaneously at the C_2 principle."
7. http://www.differencebetween.net/business/accounting-business/difference-between-gaap-and-aasb/Difference between GAAP and AASB
8. http://infomory.com/business/difference-gaap-aasb/ Difference Between GAAP and AASB\
9. http://ebooks.narotama.ac.id/files/Accounting%20for%20Managers/Chapter%202%20%20%20Generally%20Accepted%20Accounting%20Principles.pdf Generally Accepted Accounting Principles
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